Mathematical Logics
First Order Logic*

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Syntax of FOL

The alphabet of FOL is composed of two sets of symbols:

Logical symbols

- the logical constant $\bot$
- propositional logical connectives $\wedge, \vee, \supset, \neg, \equiv$
- the quantifiers $\forall, \exists$
- an infinite set of variable symbols $x_1, x_2, \ldots$
- the equality symbol $=$. (optional)

Non Logical symbols

- a set $c_1, c_2, \ldots$ of constant symbols
- a set $f_1, f_2, \ldots$ of functional symbols each of which is associated with its arity (i.e., number of arguments)
- a set $P_1, P_2, \ldots$ of relational symbols each of which is associated with its arity (i.e., number of arguments)

Non logical symbols - Example

Non logical symbols depends from the domain we want to model. Their must have an intuitive interpretation on such a domain.

Example (Domain of arithmetics)

symbols	type	arity	intuitive interpretation
0	constant	0*	the smallest natural number
$succ(\cdot)$	function	1	the function that given a number returns its successor
$+(\cdot,\cdot)$	function	2	the function that given two numbers returns the number corresponding to the sum of the two
$<(\cdot,\cdot)$	relation	2	the less then relation between natural numbers

*A constant can be considered as a function with arity equal to 0

Non logical symbols - Example

Example (Domain of arithmetics - extended)

The basic language of arithmetics can be extended with further symbols e.g:

symbols	type	arity	intuitive interpretation
0	constant	0	the smallest natural number
$succ(\cdot)$	function	1	the function that given a number returns its successor
$+(\cdot,\cdot)$	function	2	the function that given two numbers returns the number corresponding to the sum of the two
$*(\cdot,\cdot)$	function	2	the function that given two numbers returns the number corresponding to the product of the two
$<(\cdot,\cdot)$	relation	2	the less then relation between natural numbers
$\leq(\cdot,\cdot)$	relation	2	the less then or equal relation between natural numbers

Non logical symbols - Example

Example (Domain of strings)

symbols	type	arity	intuitive interpretation
E	constant	0	The empty string
“a”,“b”,	constants	0	The strings containing one single character of the latin alphabet
$concat(\cdot, \cdot)$	function	2	the function that given two strings returns the string which is the concatenation of the two
$subst(\cdot, \cdot, \cdot)$	function	3	The function that replaces all the occurrence of a string with another string in a third one
$<$	relation	2	Alphabetic order on the strings
$substring(\cdot, \cdot)$	relation	2	a relation that states if a string is contained in another string

Terms and formulas of FOL

The language of FOL consists of terms and formulas:

Terms

- every constant c_i and every variable x_i is a term;
- if $t_1, \dots, t_n$ are terms and f_i is a functional symbol of arity equal to n, then $f(t_1, \dots, t_n)$ is a term

Well formed formulas

- if t_1 and t_2 are terms then $t_1 = t_2$ is a formula
- If $t_1, \dots, t_n$ are terms and P_i is relational symbol of arity equal to n, then $P_i(t_1, \dots, t_n)$ is formula
- if A and B are formulas then $\bot$, $A \wedge B$, $A \supset B$, $A \vee B$, $\neg A$, $A \equiv B$ are formulas
- if A is a formula and x a variable, then $\forall x A$ and $\exists x.A$ are formulas.

Examples of terms and formulas

Example (Terms)

- x_i,
- c_i,
- $f_i(x_j, c_k)$, and
- $f(g(x, y), h(x, y, z), y)$

Example (formulas)

- $f(a, b) = c$,
- $P(c_1)$,
- $\exists x(A(x) \vee B(y))$,
- $P(x) \supset \exists y.Q(x, y)$.

An example of representation in FOL

Example (Language)

constants	functions (arity)	Predicate (arity)
Aldo	mark (2)	attend (2)
Bruno	best-friend (1)	friend (2)
Carlo		student (1)
MathLogic		course (1)
DataBase		less-than (2)
0, 1, . . . , 10		

Example (Terms)

Intuitive meaning	term
an individual named Aldo	Aldo
the mark 1	1
Bruno's best friend	best-friend(Bruno)
anything	x
Bruno's mark in MathLogic	mark(Bruno,MathLogic)
somebody's mark in DataBase	mark(x,DataBase)
Bruno's best friend mark in MathLogic	mark(best-friend(Bruno),MathLogic)

An example of representation in FOL (cont'd)

Example (Formulas)

Intuitive meaning	**Formula**
Aldo and Bruno are the same person	$Aldo = Bruno$
Carlo is a person and MathLogic is a course	$person(Carlo) \wedge course(MathLogic)$
Aldo attends MathLogic	$attend(Aldo, MathLogic)$
Courses are attended only by students	$\forall x(attend(x, y) \supset course(y) \supset student(x))$
	~~$\forall x(attend(x, y) \wedge course(y) \supset student(x))$~~
every course is attended by somebody	$\forall x\ (course(x) \supset \exists y\ attend(y, x))$
every student attends something	$\forall x(student(x) \supset \exists y\ attend(x, y))$
a student who attends all the courses	$\exists x(student(x) \wedge \forall y(course(y) \supset attend(x, y)))$
every course has at least two attenders	$\forall x(course(x) \supset \exists y \exists z(attend(y, x) \wedge attend(z, x) \wedge \neg y = z))$
Aldo's best friend attend the same courses attended by Aldo	$\forall x(attend(Aldo, x) \supset attend(best\text{-}friend(Aldo), x))$
best-friend is symmetric	$\forall x(best\text{-}friend(best\text{-}friend(x)) = x)$
Aldo and his best friend have the same mark in MathLogic	$mark(\text{best-friend}(Aldo), MathLogic) = mark(Aldo, MathLogic)$
A student can attend at most two courses	$\forall x \forall y \forall z \forall w(attend(x, y) \wedge attend(x, z) \wedge attend(x, w) \supset (y = z \vee z = w \vee y = w))$

Common Mistakes

- **Use of $\wedge$ with $\forall$**

  $\forall x\, (WorksAt(FBK, x) \wedge Smart(x))$ means "Everyone works at FBK and everyone is smart"

  "Everyone working at FBK is smart" is formalized as

  $$\forall x\, (WorksAt(FBK, x) \supset Smart(x))$$

- **Use of $\supset$ with $\exists$**

  $\exists x\, (WorksAt(FBK, x) \supset Smart(x))$ means "There is a person so that if (s)he works at FBK then (s)he is smart" and this is true as soon as there is at last an x who does not work at FBK

  "There is an FBK-working smart person" is formalized as

  $$\exists x\, (WorksAt(FBK, x) \wedge Smart(x))$$

Representing variations quantifiers in FOL

Example

Represent the statement at least 2 students attend the KR course

$$\exists x_1 \exists x_2 (attend(x_1, KR) \wedge attend(x_2, KR))$$

The above representation is not enough, as x_1 and x_2 are variable and they could denote the same individual, we have to guarantee the fact that x_1 and x_2 denote different person. The correct formalization is:

$$\exists x_1 \exists x_2 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge x_1 \neq x_2)$$

At least n . . .

$$\exists x_1 \ldots x_n \left(\bigwedge_{i=1}^{n} \varphi(x_i) \wedge \bigwedge_{i \neq j = 1}^{n} x_j \neq x_i \right)$$

Representing variations of quantifiers in FOL

Example

Represent the statement at most 2 students attend the KR course

$$\forall x_1 \forall x_2 \forall x_3 (attend(x_1, KR) \wedge attend(x_2, KR) \wedge attend(x_3, KR) \supset x_1 = x_2 \vee x_2 = x_3 \vee x_1 = x_3)$$

At most n . . .

$$\forall x_1 \ldots x_n \ (\bigwedge_{i=1}^{n} \varphi(x_i) \ \supset \ \bigwedge_{i \neq j=1}^{n} x_j = x_i)$$

Errata – corridge: il simbolo di <u>congiunzione</u> dopo implicazione in equazione sopra va letto come <u>disgiunzione</u> . Si confronti con esempio

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