Mathematical Logic
PL - Reasoning as deduction

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Lecture index

1. Recap of basic notions
2. Reasoning as deduction
3. Hilbert systems (VAL – forward chaining)
4. Tableaux systems ((un)-SAT – backward chaining)

Reasoning / Decision problems

Not via deduction

Four tipes of questions

- ~~**Model Checking – MC** (I, φ): $I \overset{?}{\models} \varphi$.
  What is the truth value of φ in I, or equivalently, does I satisfy φ or does it not satisfy φ.~~

- **(Un)Satisfiability – SAT/ UNSAT** (φ): $\exists I \,.\, I \overset{?}{\models} \varphi$
  Is there a model I that satisfies φ?

- **Validity - VAL**(φ): $\overset{?}{\models} \varphi$. Is φ satisfied by all the models I ?

- **Logical consequence**(Γ, φ): $\Gamma \overset{?}{\models} \varphi$
  Is φ satisfied by all the models I that satisfy all the formulas in Γ?

Reminder

Proposition

A *Valid* $\longrightarrow A$ *satisfiable* $\longleftrightarrow A$ *not unsatisfiable*

A *unsatisfiable* $\longleftrightarrow A$ *not satisfiable* $\longrightarrow A$ *not Valid*

$\Gamma, A \vDash B \longleftrightarrow \Gamma \vDash A \rightarrow B$

$\Gamma \vDash \varphi \longleftrightarrow \Gamma \cup \{\neg\varphi\}$ *not satisfiable*

Proposition

if A is	*then ¬A is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

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Deduction / Proof

Given

1. *Premises:* Γ
2. *Conclusion:* A

A deduction /proof is Sequence, Tree/ Direct Acyclic Graph (DAG) of nodes, where

- *Each node of the deduction labeled with a formula*
- *Links labeled with motivation (so called «inference rules»)*
- *Root nodes are premises*
- *Leaf node(s) is conclusion*

We write $\Gamma \vdash A$ – to mean that there is (at least a) deduction which «connects» Γ and A.

Key properties that we want satisfied: Correctness theorem ($\Rightarrow$) and Completeness theorem ($\Leftarrow$), in formulas:

$$\Gamma \vdash \varphi \quad \textit{iff} \quad \Gamma \vDash \varphi$$

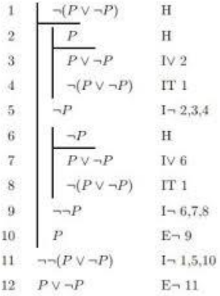
NOTE: computation of other logical properties listed in Recap follows.

Deductions (examples)

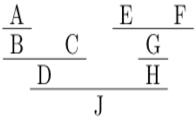
Examples of deductions (as defined by different logics)

1. Example 1: Sequence
2. Example 2: Forward Tree/ Direct Acyclic Graph (DAG)
3. Example 3: Backward Tree/ DAG

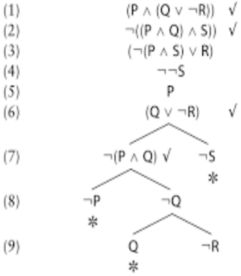
(example 1)



(example 2)



(example 3)



Inference rules (examples)

Rules of Inference

Modus Ponens

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Addition

$$\begin{array}{l} p \\ \hline p \vee q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

Types of Deductions

Two types of deductions (as defined by different logics)

1. Forward deductions (generate theorems from theorems)
2. Backward deducions (generate subgoals from goals)

Forward deductions (as defined by logics with forward calculus):

- Good for proving properties of logics
- Bad for deriving consequences (reasoning) of what is known
- Used in mathematical logics

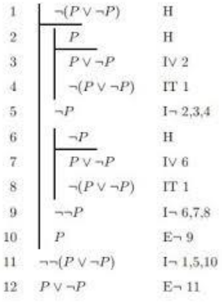
Backward deductions (as defined by logics with forward calculus):

- Good for reasoning
- A little harder for proving properties of logics
- Used in Computer Science/ Artificial Intelligence

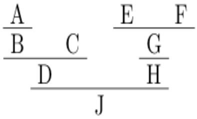
Forward Deduction (examples)

- Premises: what is known or assumed (*axioms* or *assumptions*)
- Conclusions: what we want to discover (*theorems/ goals*)
- Shape: (Forward path) or Forward Tree/ DAG
- Problem: how do you know where to go? Search motivated by goal.

(example 1)



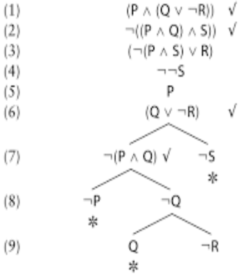
(example 2)



Backward Deduction (examples)

- Premises: the *goal* to be proved
- Conclusions: some *termination condition* which guarantees that the goal derives from what is known (i.e., it is a theorem)
- Shape: Backward DAG
- Problem: In which direction to expand the proof, given exponential blow up (need very complex heuristics)

(example 3)



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