Mathematical Logics
Propositional Logic *

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Reference(s):

- **Francesco Berto, Logica da zero a Gödel, Laterza, 2018 (capitolo 1)**

**Originally by Luciano Serafini and Chiara Ghidini*
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Propositional theory

Definition (Propositional theory)

A theory is a set of formulas closed under the logical consequence relation. I.e. T is a theory iff $T \models A$ implies that $A \in T$

Example (Of theory)

- T_1 is the set of valid formulas $\{A | A \text{ is valid}\}$
- T_2 is the set of formulas which are true in the interpretation $I = \{P, Q, R\}$
- T_3 is the set of formulas which are true in the set of interpretations $\{I_1, I_2, I_3\}$
- T_4 is the set of all formulas

Show that T_1, T_2, T_3 and T_4 are theories

Propositional theory (2)

Example (Of non theory)

- N_1 is the set $\{A,\ A \longrightarrow B,\ C\}$
- N_2 is the set $\{A, A \longrightarrow B, B, C\}$
- N_3 is the set of all formulas containing P

Show that N_1, N_2 and N_3 are not theories

Axiomatization

Remark

A propositional theory always contains an infinite set of formulas. Indeed any theory T contains at least all the valid formulas. which are infinite) (e.g., $A \rightarrow A$ for all formulas A)

Definition (Set of axioms for a theory)

A set of formulas Ω is a set of axioms for a theory T if for all $A \in T$, $\Omega \models A$.

Definition (Finitely axiomatizable theory)

A theory T is finitely axiomatizable if it has a finite set of axioms.

Propositional theory (cont'd)

Definition (Logical closure)

For any set Γ, $cl(\Gamma) = \{A \mid \Gamma \models A\}$

Proposition (Logical closure)

For any set Γ, the logical closure of Γ, $cl(\Gamma)$ is a theory

Proposition

Γ is a set of axioms for $cl(\Gamma)$. (NOTE: not the only one)

Example: Hilbert axioms for propositional logic

A1 $\varphi \supset (\psi \supset \varphi)$

A2 $(\varphi \supset (\psi \supset \vartheta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \vartheta))$

A3 $(\neg\psi \supset \neg\varphi) \supset ((\neg\psi \supset \varphi) \supset \varphi)$

NOTE:

1. Minimal language (extra logical connectives defined in terms of the basic ones)
2. It allows to compute all tautologies
3. Useful to prove properties of logical theories (e.g., correctness and completeness of mathematics)
4. Never used in practice in CS (full use of connectives and more)
5. In practice in CS, people add more axioms (facts) which define what is true in the intended (mental) model

Using logics in Practice – Reminder from lecture 1

- Define a logic
  - most often by reseachers (almost never in real world applications)
  - Done once for all (very hard. Examples. PL, FOL, DL, ML, …)
- Choose the right logic for the problem (L, I, ⊨)
  - Given a problem the computer scientist must choose the right logic, most often one of the many available
- Write the theory
  - The computer scientist writes a theory T (making sure it complies to intended mental model)
- Different uses (see above)
  - Use theory as basis for agreement developer/customer
  - Use theory to guarantee semantic interoperability (e.g., as in data integration, program composition, …)
  - Use reasoning (entailment) to make sure program does what it is supposed to do
  - Use reasoning to implement AI
  - …

Axioms and theory - intuition

Compact representation of knowledge

The axiomatization of a theory is a compact way to represent a set of interpretations, and thus to represent a set of possible (acceptable) world states. In other words is a way to represent all the knowledge we have of the real world.

Minimality

The axioms of a theory constitute the basic knowledge, and all the *knolwledge* can be obtained by logical consequence.

An important feature of a set of axioms, is that they are minimal, i.e., no axioms can be derived from the others.

Axioms and theory - intuition

Example

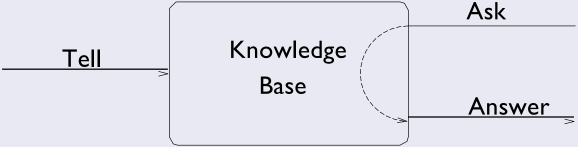
$Pam\_Attends\_Logic\_Course$
$John\_is\_a\_Phd\ Student$
$Pam\_Attends\_Logic\_Course \rightarrow Pam\_is\_a\_Ms\_Student \vee Pam\_is\_a\_PhD\_Student$
$Pam\_is\_a\_Ms\_Student \rightarrow \neg Pam\_is\_a\_Ba\_Student$
$Pam\_is\_a\_PhD\_Student \rightarrow \neg Pam\_is\_a\_Ba\_Student$
$\neg (John\_is\_a\_Phd\_Student \wedge John\_is\_a\_Ba\_Student)$

The axioms above constitute the basic knowledge about the people that attend logic course. The facts $\neg Pam\_is\_a\_Ba\_Student$ and $\neg John\_is\_a\_Ba\_Student$ don't need to be added to this basic knowledge, as they can be derived via logical consequence.

Logic based systems

A logic-based system for representing and reasoning about knowledge is composed by a Knowledge base and a Reasoning system. A knowledge base consists of a finite collection of formulas in a logical language. The main task of the knowledge base is to answer queries which are submitted to it by means of a Reasoning system

Logic based system for knowledge representation



Tell: this action incorporates the new knowledge encoded in an axiom (formula). This allows to build a *KB*.

Ask: allows to query what is known, i.e., whether a formula ϕ is a logical consequences of the axioms contained in the KB ($KB \models \phi$)

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