Mathematical Logics
Propositional Logic (Properties and proofs) [Optional]

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The content of these slides is optional – not tested by the exam

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Properties of propositional logical consequence

Proposition

If Γ and Σ are two sets of propositional formulas and A and B two formulas, then the following properties hold:

- Reflexivity $\{A\} \models A$
- Monotonicity *If* $\Gamma \models A$ *then* $\Gamma \cup \Sigma \models A$
- Cut *If* $\Gamma \models A$ *and* $\Sigma \cup \{A\} \models B$ *then* $\Gamma \cup \Sigma \models B$
- Compactness *If* $\Gamma \models A$*, then there is a finite subset* $\Gamma_0 \subseteq \Gamma$*, such that* $\Gamma_0 \models A$
- Deduction theorem *If* $\Gamma, A \models B$ *then* $\Gamma \models A \rightarrow B$
- Refutation principle $\Gamma \models A$ *iff* $\Gamma \cup \{\neg A\}$ *is unsatisfiable*

NOTE: *vice versa of deduction theorem trivial*

Properties of propositional logical consequence

Reflexivity $\{A\} \vDash A$.

PROOF: For all I if $I \vDash A$, then $I \vDash A$.

Monotonicity **If $\Gamma \vDash A$ then $\Gamma \cup \Sigma \vDash A$**

PROOF: For all I, if $I \vDash \Gamma \cup \Sigma$, then $I \vDash \Gamma$. Then by hypothesis ($\Gamma \vDash A$) we can infer that $I \vDash A$, and therefore that $\Gamma \cup \Sigma \vDash A$

Properties of propositional logical consequence

Cut **If $\Gamma \vDash A$ and $\Sigma \cup \{A\} \vDash B$ then $\Gamma \cup \Sigma \vDash B$.**

PROOF:

(i) In the premise of the conclusion of the theorem, for the definition of consequence relation, we have that, for all I, if $I \vDash \Gamma \cup \Sigma$, then $I \vDash \Gamma$ and $I \vDash \Sigma$.

(ii) The first theorem hypothesis $\Gamma \vDash A$ implies that if $I \vDash \Gamma$ then $I \vDash A$, namely, from (i), $I \vDash A$.

(iii) Since from (i) we have that $I \vDash \Sigma$, then from (ii) $I \vDash \Sigma \cup \{A\}$.

(iv) The second theorem hypothesis $\Sigma \cup \{A\} \vDash B$, implies that $I \vDash B$.

(v)We can therefore conclude, from (iii) and (iv), that $\Gamma \cup \Sigma \vDash B$.

Properties of propositional logical consequence

Compactness **If $\Gamma \vDash A$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$, such that $\Gamma_0 \vDash A$.**

(REMEMBER: (o) A formula *A* is a logical consequence of a set of formulas Γ, in symbols $\Gamma \vDash A$ iff any interpretation *I* that satisfies all the formulas in Γ satisfies also *A*)

PROOF:

(i) Trivial if Γ is finite. Trivial if A is a tautology. Assume A and therefore Γ satisfiable. Let us consider infinite case with A not a tautology.

(ii) Let PA be the set of primitive propositions occurring in A (PA finite, being A one formula).

(iii) Let $I_1, \ldots, I_n$ (with $n \leq 2^{|PA|}$, n finite), be all the interpretations I_i of PA that do not satisfy A, namely $I_i \nvDash A$. They must exist as A is not a tautology.

(iv) From $\Gamma \vDash A$ then there should be $I'_1, \ldots, I'_n$ interpretations of the language of Γ, which are extensions of $I_1, \ldots, I_n$, and such that $I' \nvDash \Gamma_k$ for some $\Gamma_k \in \Gamma$ (from (iii) and (o): if conclusion of implication does not hold then the premise does not hold).

(v) Let $\Gamma_0 = \{\Gamma_1, \ldots, \Gamma_k\}$. Then $\Gamma_0 \vDash A$ (vacuously true since premise is false).

(vi) Indeed if $I \vDash \Gamma_0$ then I is an extension of an interpretation J of PA that satisfies A, and therefore $I \vDash A$.

Properties of propositional logical consequence

Deduction theorem **If $\Gamma, A \vDash B$ then $\Gamma \vDash A \rightarrow B$**

PROOF:

(1) Assume by hypothesis that $I \vDash \Gamma$. We have two cases:

(1.1) If $I \vDash A$, then $I \vDash B$ from hypothesis and therefore $I \vDash A \rightarrow B$.

(see inductive definition of implication satisfiability, i.e., $I \vDash A \rightarrow B$ when $I \vDash A$ then $I \vDash B$)

(1.2) If $I \nvDash A$, then (**false**) $\vDash B$ from hypothesis (since from (1) $I \vDash \Gamma$), and therefore

$I \vDash A \rightarrow B$ *(in the hypothesis, if for every I the premise is false the implication is always true)*

(2) We can therefore conclude that $I \vDash A \rightarrow B$.

Refutation principle $\Gamma \models A$ iff $\Gamma \cup \{\neg A\}$ is unsatisfiable

PROOF:

(⇒)

(i) Suppose by contradiction that $\Gamma \cup \{\neg A\}$ is satisfiable.

(ii) This implies that there is an interpretation I such that $I \models \Gamma$ and $I \models \neg A$, i.e., $I \not\models A$.

(iii) This contradicts that fact (stated in the hypothesis) that all interpretations that satisfy Γ also satisfy A

(⇐)

(i) Let $I \models \Gamma$.

(ii) Then by the fact that $\Gamma \cup \{\neg A\}$ is unsatisfiable, we have that $I \not\models \neg A$,

(iii) Therefore $I \models A$.

(iv) We can conclude that $\Gamma \models A$ *(iff for all I, both $I \models \Gamma$ and $I \models A$, then $\Gamma \models A$)*

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