Mathematical Logics
Propositional Logic *

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Reference(s):

- **Francesco Berto, Logica da zero a Gödel, Laterza, 2018 (capitolo 1)**

**Originally by Luciano Serafini and Chiara Ghidini*
Modified by Fausto Giunchiglia and Mattia Fumagalli

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Interpretation of Propositional Logic

Definition (Interpretation)

A Propositional interpretation is a function $I : \mathbf{PROP} \rightarrow \{\text{True}, \text{False}\}$

Remark

If $|\mathbf{PROP}|$ is the cardinality of $\mathbf{PROP}$, then there are 2** $|\mathbf{PROP}|$ different interpretations, i.e. all the different subsets of $\mathbf{PROP}$.

If $|\mathbf{PROP}|$ is finite then there is a finite number of interpretations.

Remark

A propositional interpretation can be thought as a subset S of $\mathbf{PROP}$, and I is the characteristic function of S, i.e.,

$$A \in S \text{ iff } I(A) = \text{True}.$$

Interpretation of Propositional Logic

Example

	p	q	r	Set theoretic representation
I_1	True	True	True	$\{p, q, r\}$
I_2	True	True	False	$\{p, q\}$
I_3	True	False	True	$\{p, r\}$
I_4	True	False	False	$\{p\}$
I_5	False	True	True	$\{q, r\}$
I_6	False	True	False	$\{q\}$
I_7	False	False	True	$\{r\}$
I_8	False	False	False	$\{\}$

Satisfiability of a propositional formula

Definition (I satisfies a formula, $I \models A$)

A formula A is true in / satisfied by an interpretation I, in symbols $I \models A$, according to the following inductive definition:

- $I \models P$ if $I(P) = \text{True}$, with $P \in \mathbf{PROP}$
- $I \models \neg A$ if not $I \models A$
- $I \models A \wedge B$ if, $I \models A$ and $I \models B$
- $I \models A \vee B$ if, $I \models A$ or $I \models B$
- $I \models A \rightarrow B$ if, when $I \models A$ then $I \models B$
- $I \models A \equiv B$ if, $I \models A$ iff $I \models B$

Satisfiability of a propositional formula

Example (interpretation)

Let $\mathcal{P} = \{P, Q\}$.
$\mathcal{I}(P) = True$ and $\mathcal{I}(Q) = False$ can be also expressed with
$\mathcal{I} = \{P\}$.

Example (Satisfiability)

Let $\mathcal{I} = \{P\}$. Check if $\mathcal{I} \models (P \wedge Q) \vee (R \rightarrow S)$:
Replace each occurrence of each primitive propositions of the formula with the truth value assigned by $\mathcal{I}$, and apply the definition for connectives.

$$(\text{True} \wedge \text{False}) \vee (False \rightarrow \text{False}) \tag{1}$$

$$\text{False} \vee \text{True} \tag{2}$$

$$\text{True} \tag{3}$$

Satisfiability of a propositional formula

Proposition

If for any propositional variable P appearing in a formula A,
$I(P) = I'(P)$, *then* $I \models A$ *iff* $I' \models A$

Checking if $I \models A$

Lazy evaluation algorithm (1/2)

$(A = p)$

check($I \models p$)
 if $I(p) = True$
  then return YES
 else return NO

$(A = B \wedge C)$

check($I \models B \wedge C$) if
 check($I \models B$)
  then return check($I \models C$)
 else return NO

$(A = B \vee C)$

check($I \models B \vee C$) if
 check($I \models B$)
  then return YES
 else return check($I \models C$)

Checking if $I \models A$

Lazy evaluation algorithm (2/2)

$(A = B \rightarrow C)$

check($I \models B \rightarrow C$) if
    check($I \models B$)
        then return check($I \models C$)
    else return YES

$(A = B \equiv C)$

check($I \models B \equiv C$) if
    check($I \models B$)
        then return check($I \models C$)
    else return not(check($I \models C$)

Formalizing English Sentences

Exercise

Let's consider a propositional language where p means *"Paola is happy"*, q means *"Paola paints a picture"*, and r means *"Renzo is happy"*.
Formalize the following sentences:

1. *"if Paola is happy and paints a picture then Renzo isn't happy"*
   $p \wedge q \rightarrow \neg r$
2. *"if Paola is happy, then she paints a picture"*
   $p \rightarrow q$
3. *"It is not the case that Paola is happy and she does not paint a picture"*
   $\neg (p \wedge \neg q)$ which is equivalent to $p \rightarrow q$!!!

The precision of formal languages allows to avoid the ambiguities of natural languages.

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