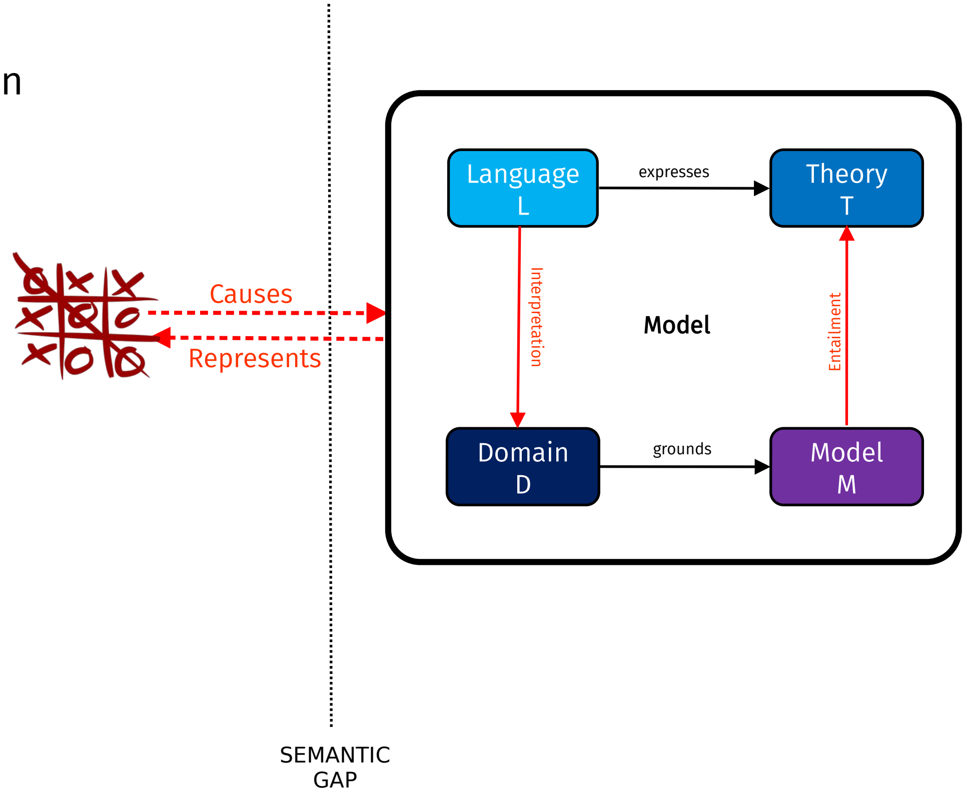
Computational Logic
L4.X.1B Exercises

Adolfo Villafiorita
Fausto Giunchiglia

UNIVERSITY OF TRENTO

Modeling Tic-tac-toe: hints

- How can you represent the board and pieces on the board?
- What is the domain?
- How does the language look like? Try first with a language with symbols only; then with symbols and rules.
- What is the interpretation function?
- How does a model in which none wins look like?
- What is the theory in which crosses win? What is the theory in which naughts win?



Preliminary Considerations

New!

- There are different ways and different models (= L, D, T, M) which represent the same world
- Each model is a "comprehensible" representation of the world once one fills the **semantic gap**, that is, the modeler explains in which way the model represents the world
- The modeler typically chooses a model fulfilling one or more specific goals (for instance, the most natural representation, the most efficient from the computational point of view)
- The modeling activity is often iterative: the modeler tries different approaches and, once a satisfactory approach is found, the modeler refines the models different times before starting to work with it

Modeling Tic-Tac-Toe

New!

- In the following we present two different models for the Tic-tac-toe game
- Not all details are explained, but you should get the overall sense and approach
- Each model has got its own advantages and disadvantages

Approach 1:
starting from the domain,
one symbol per board

New!

Domain: by enumeration

O	X	
	O	
		X

- We use a set to represent the domain
- Each element of the domain represents the status of a board, that is, where Xs and Os are

Counting on base N (recap)

New!

Base 2

0..1

	2^8							2^1	2^0
...	1	...					...	0	1

$= 1 * 2^0 + 0 * 2^1 + \ldots + 1*2^8 + \ldots$

Base 3

0..2

	3^8							3^1	3^0
...	0	...					...	2	1

$= 1 * 3^0 + 2 * 3^1 + \ldots + 0*3^8 + \ldots$

Base N

0..(N-1)

	N^8							N^1	N^0
...	1	...					...	4	1

$= 1 * N^0 + 4 * N^1 + \ldots + 1*N^8 + \ldots$

Domain: by enumeration

9	...	
		1

POSITION

ENCODING:

	= 0
o	= 1
x	= 2

3^8							3^1	3^0
9	...					...	2	1

$$D = \{x | x \in \mathbb{N} \wedge x <= 3^9 - 1\}$$

notice that this domain contains board configurations that are not reachable in the game of Tic Tac Toe, e.g., a board with all Xs

Example: board 332

New!

9	8	7
6 O	5 O	4
3	2 X	1 X

9	8	7	6	5	4	3	2	1
			O	O			X	X

	= 0
o	= 1
x	= 2

3^8	3^7	3^6	3^5	3^4	3^3	3^2	3^1	3^0
			1	1			2	2

$$= 2 * 3^0 + 2 * 3^1 + 1 * 3^4 + 1 * 3^5$$
$$= 332$$

Language: by Enumeration

Revised!

- **One propositional symbol per board status: B_0, B_1, B_3, B_4, ...**
- However, what does **B_0** mean?
- This is determined by the **interpretation function**
- For instance, if we decide that **$I(B_0)$ = 0**, then we are saying that the meaning of **B_0 (language) is 0 (domain), which corresponds to the empty board (semantic gap)**

- However, many symbols, we need:
  - Khmer alphabet (the largest alphabet in the world), consists of 33 consonants, 23 vowels and 12 independent vowels
  - The total number of Chinese characters ever to appear in a dictionary is in the tens of thousands (of which about 3000-4000 are known to a college graduate)

Representing Winning Positions

Revi
sed!

- The winning positions W_x is the subset of D which includes all board positions corresponding to a victory for X, e.g.:
  - All Xs on lowest row: $2 * 3^0 + 2 * 3^1 + 2 * 3^2 = 26$
  - All Xs on middle row: $2 * 3^3 + 2 * 3^4 + 2 * 3^5 = 702$
  - …
  - All Xs on lowest row, one “O” in position 5 and another “O” in position 6
  - …
- Hence: $W_x = \{ 26, 702, \dots \}$

(Notice that the first two “winning” boards we have shown above correspond to configurations which are not legal in the game of Tic Tac Toe, since there are no Os.)

Representing Winning Positions

Revi
sed!

- Suppose now that we add a symbol **V_x** to our language which represents the winning conditions for X
- Then: **I(V_x) =** $\mathbf{W}_x$

Issues with this model

- Difficult to manage: e.g. what board corresponds to $\mathbf{B}_{42}$ (language) or **42** (domain)?
- Difficult to use: how do we represent the fact that some board configurations are not possible, in a fair game?

Approach 2:
start from the language, one symbol per cell

UNIVERSITY
OF TRENTO

Language (propositional logic)

- One symbol per **cell** status
- Boards represented as propositional formulas

C_1 = cross in square 1

…

N_1 = naught in square 1

…

Domain and Interpretation Function

Revised!

$D = \{ T, F \}$

$I : L \rightarrow D$

- Domain: truth values
- The interpretation function maps the propositional letters of the language to T or F.
- For instance, if $\mathbf{I(C_1) = T}$ means that $\mathbf{C_1}$ is true, that is, *square 1 contains an "X" (semantic gap).*

Using formulas for the board

Revi
sed!

<table>
<tr><th>World</th><th>Corresponding representation in the language</th><th>Interpretation function will certainly have …</th></tr>
<tr>
<td>
<table>
<tr><td>9 o</td><td></td><td></td></tr>
<tr><td></td><td>o</td><td></td></tr>
<tr><td>x</td><td>x</td><td>1 x</td></tr>
</table>
</td>
<td>$C_1 \wedge C_2 \wedge C_3 \wedge N_5 \wedge N_9$</td>
<td>$I(C_1) = T$
$I(C_2) = T$
$I(C_3) = T$
$I(N_5) = T$
$I(N_9) = T$</td>
</tr>
<tr>
<td>
<table>
<tr><td>9 o</td><td>x</td><td></td></tr>
<tr><td></td><td>o</td><td></td></tr>
<tr><td>x</td><td></td><td>1 o</td></tr>
</table>
</td>
<td>$C_3 \wedge C_8 \wedge N_1 \wedge N_5 \wedge N_9$</td>
<td>$I(C_3) = T$
$I(C_8) = T$
$I(N_1) = T$
$I(N_5) = T$
$I(N_9) = T$</td>
</tr>
</table>

Starting from the Language

- Suppose we now state the following formula to be true:

$$C_1 \wedge C_2 \wedge C_3 \wedge N_5 \wedge N_9$$

  (that is C_1 is true and C_2 is true, …)

- What can we say about a symbol not appearing in the formula, e.g., C_4? **Nothing, of course.** Its interpretation is not defined by the formula and it might be true or false.

Starting from the language

New!

- There are many different interpretation functions which are “compatible” with the formula, namely all the functions which assign any value to the letters not appearing in the formula true, such as, for instance, I_1 and I_2 (see right)
- Notice that this corresponds to saying that the formula of the previous slide represents (or “is compatible with”) many different boards, that is all the boards where the content of the cells which are not mentioned is empty, “o”, or “x”

$$C_1 \wedge C_2 \wedge C_3 \wedge N_5 \wedge N_9$$

9 o	?	?
?	o	?
x	x	1 x

$I_1(C_1) = T$
$I_1(C_2) = T$
$I_1(C_3) = T$
$I_1(C_4) = F$
$I_1(N_5) = T$
$I_1(N_9) = T$
…

$I_2(C_1) = T$
$I_2(C_2) = T$
$I_2(C_3) = T$
$I_2(C_4) = T$
$I_2(N_5) = T$
$I_2(N_9) = T$
…

Using the Theory

- The theory selects the interpretation functions we are interested in

Asserting C_1 in the theory “eliminates” from the model all interpretation functions such that $I(C_1) = F$

Representing victories for Cross

9	8	7
6	5	4
3	2	1

$$(C_1 \land C_2 \land C_3) \lor (C_4 \land C_5 \land C_6) \lor (C_7 \land C_8 \land C_9) \lor$$
$$(C_1 \land C_4 \land C_7) \lor (C_2 \land C_5 \land C_8) \lor (C_3 \land C_6 \land C_9) \lor$$
$$(C_3 \land C_5 \land C_7) \lor (C_1 \land C_5 \land C_9)$$

Notice that this formula alone asserts victory conditions for X, but it is true also in models in which both X and O win, in which X and O can occupy the same squares at the same "time", and in which X wins in boards not resulting from a fair game (e.g. all Xs and no Os). Notice also that C_1 might be interpreted as "naught in square 1", "reversing" the meaning we give to all formulas

Refining the Theory

- In our formalization, which formula(s) represent the fact that the same square cannot be occupied at the same time by a X and a O?
- Can you easily represent a game in propositional logic?

Questions:

https://github.com/avillafiorita/cl-2020

Adolfo Villafiorita
Fausto Giunchiglia

UNIVERSITY
OF TRENTO