Mathematical Logics
Set Theory*

Fausto Giunchiglia and Mattia Fumagallli

University of Trento

**Originally by Luciano Serafini and Chiara Ghidini*
Modified by Fausto Giunchiglia and Mattia Fumagalli

Outline

1. Introduction and motivation
2. Basic notions
3. Relations
4. Functions
5. Exercises

Relations

- A relation R from the set A to the set B is a subset of the Cartesian product of A and B: $R \subseteq A \times B$; if $(x,y) \in R$, then we will write xRy for 'x is R-related to y'.

- A binary relation on a set A is a subset $R \subseteq A \times A$

- **Examples**:

  - given $A = \{1, 2, 3, 4\}$, $B = \{a, b, d, e, r, t\}$ and aRb iff in the Italian name of a there is the letter b, then $R = \{(2, d), (2, e), (3, e), (3, r), (3, t), (4, a), (4, r), (4, t)\}$

  - given $A = \{3, 5, 7\}$, $B = \{2, 4, 6, 8, 10, 12\}$ and aRb iff a is a divisor of b, then $R = \{(3, 6), (3, 12), (5, 10)\}$

- **Exercise**: in prev. example, let aRb iff $a + b$ is an even number
  $R = ?$

Relations (2)

- Given a relation R from A to B,
  - the domain of R is the set $Dom(R) = \{a \in A \mid \text{there exists a } b \in B, aRb\}$
  - the co-domain of R is the set $Cod(R) = \{b \in B \mid \text{there exists an } a \in A, aRb\}$

- Let R be a relation from A to B. The inverse relation of R is the relation $R^{-1} \subseteq B \times A$ where
  $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Relation Properties

- Let R be a binary relation on A. R is
  - reflexive iff aRa for all $a \in A$;
  - symmetric iff aRb implies bRa for all $a, b \in A$;
  - transitive iff aRb and bRc imply aRc for all $a, b, c \in A$;
  - anti-symmetric iff aRb and bRa imply $a = b$ for all $a, b \in A$;

Equivalence

- Let R be a binary relation on a set A. R is an equivalence relation iff it satisfies all the following properties:
  - reflexive
  - symmetric
  - transitive
- an equivalence relation is usually denoted with $\sim$ or $\equiv$

Set Partition

- Let A be a set, a partition of A is a family F of non-empty subsets of A s.t.:
  - the subsets are pairwise disjoint
  - the union of all the subsets is the set A
- Notice that: each element of A belongs to exactly one subset in F.

Order Relation

- Let A be a set and R be a binary relation on A. R is an order (partial), usually denoted with $\leq$, if it satisfies the following properties:
  - reflexive $a \leq a$
  - anti-symmetric $a \leq b$ and $b \leq a$ imply $a = b$
  - transitive $a \leq b$ and $b \leq c$ imply $a \leq c$
- If the relation holds for all $a, b \in A$ then it is a total order
- A relation is a strict order, denoted with $<$, if it satisfies the following properties:
  - transitive $a < b$ and $b < c$ imply $a < c$
  - for all $a, b \in A$ either $a < b$ or $b < a$ or $a = b$

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