Mathematical Logics
Set Theory*

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Outline

The concept of **set** is considered a primitive concept in math

A set is a collection of elements whose description must be unambiguous and unique: it must be possible to decide whether an element belongs to the set or not.

Examples:

the students in this classroom , the points in a straight line , the cards in a playing pack

are all sets

Describing Sets

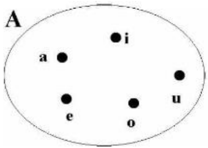
In set theory there are several description methods:

Listing: the set is described listing all its elements Example: $A = \{a, e, i, o, u\}$.

Abstraction: the set is described through a property of its elements

Example: $A = \{x \mid x$ is a vowel of the Latin alphabet $\}$

Eulero-Venn Diagrams: graphical representation that supports the formal description



Empty Set: $\emptyset$ is the set containing no elements;
Membership: $a \in A$, element belongs to the set A;

Non membership: $a \notin A$, element a doesn't belong to the set A;

Equality: $A = B$, iff the sets A and B contain the same elements;

inequality: $A \neq B$, iff it is not the case that $A = B$

Subset: $A \subseteq B$, iff all elements in A belong to B too;
Proper subset: $A \subset B$, iff $A \subseteq B$ and $A \neq B$

Power Set

We define the power set of a set A, denoted with $P(A)$, as the set containing all the subsets of A.

Example: if $A = \{a,b,c\}$, then
$P(A) = \{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\}$

If A has n elements, then its power set $P(A)$ contains 2^n elements.

Exercise: prove it!!!

Operations on Sets

Union: given two sets A and B we define the union of A and B as the set containing the elements belonging to A or to B or to both of them, and we denote it with $A \cup B$.

Example: if $A = \{a,b,c\}$, $B = \{a,d,e\}$ then $A \cup B = \{a,b,c,d,e\}$

Intersection: given two sets A and B we define the intersection of A and B as the set containing the elements that belongs both to A and B, and we denote it with $A \cap B$.

Example: if $A = \{a,b,c\}$, $B = \{a,d,e\}$ then $A \cap B = \{a\}$

Operations on Sets (2)

Difference: given two sets A and B we define the difference of A and B as the set containing all the elements which are members of A, but not members of B, and denote it with $A - B$.

Example: if $A = \{a,b,c\}$, $B = \{a,d,e\}$ then $A - B = \{b,c\}$

Complement: given a universal set U and a set A, where $A \subseteq U$, we define the complement of A in U ,denoted with $\overline{A}$ (or $C_U A$), as the set containing all the elements in U not belonging to A.

Example: if U is the set of natural numbers and A is the set of even numbers (0 included), then the complement of A in U is the set of odd numbers.

Examples:

Given $A = \{a, e, i, o, \{u\}\}$ and $B = \{i, o, u\}$, consider the following statements:

1. $B \in A$ NO!
2. $(B - \{i, o\}) \in A$ OK
3. $\{a\} \cup \{i\} \subset A$ OK
4. $\{u\} \subset A$ NO!
5. $\{\{u\}\} \subset A$ OK
6. $B - A = \varnothing$ NO! $B - A = \{u\}$
7. $i \in A \cap B$ OK
8. $\{i, o\} = A \cap B$ OK

Sets: Operation Properties

- $A \cap A = A$,
  $A \cup A = A$
- $A \cap B = B \cap A$,
  $A \cup B = B \cup A$ (*commutative*)
- $A \cap \varnothing = \varnothing$
  $A \cup \varnothing = A$
- $(A \cap B) \cap C = A \cap (B \cap C)$,
  $(A \cup B) \cup C = A \cup (B \cup C)$ (*associative*)

Sets: Operation Properties (2)

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (distributive)

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$

  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (De Morgan laws)

- **Exercise**: Prove the validity of all the properties.

Cartesian Product

Given two sets A and B, we define the Cartesian product of A and B as the set of ordered couples (a, b) where $a \in A$ and $b \in B$; formally,
$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Notice that: $A \times B \neq B \times A$

Cartesian Product (2)

- **Examples**:
  - given $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then
    $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$ and
    $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.
  - Cartesian coordinates of the points in a plane are an example of the Cartesian product R × R

- The Cartesian product can be computed on any number n of sets $A_1, A_2 \ldots, A_n$, $A_1 \times A_2 \times \ldots \times A_n$ is the set of ordered n-tuple $(x_1, \ldots, x_n)$ where $x_i \in A_i$ for each $i = 1 \ldots n$.

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