Mathematical Logics
Modal Logic: Reasoning*

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1. Introduction
2. Tableaux rules
3. Exercises (validity, satisfiability with model construction

Analytic/Semantic Tableau Method - References

Early work by Beth and Hintikka (around 1955). Later refined and popularized by Raymond Smullyan:

- R.M. Smullyan. *First-order Logic*. Springer-Verlag, 1968.

Modern expositions include:

- M. Fitting. *First-order Logic and Automated Theorem Proving. 2nd edition.* Springer-Verlag, 1996.
- M. D'Agostino, D. Gabbay, R. Hähnle, and J. Posegga (eds.). *Handbook of Tableau Methods*. Kluwer, 1999.
- R. Hähnle. *Tableaux and Related Methods*. In: A. Robinson and A. Voronkov (eds.), *Handbook of Automated Reasoning*, Elsevier Science and MIT Press, 2001.
- Proceedings of the yearly Tableaux conference: http://i12www.ira.uka.d/TABLEAUX/

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Tableau - basic definition

Definition

Tableau A tableau is a finite tree with nodes marked with one of the following assertions:

$$w \models \phi \qquad w \not\models \phi \qquad wRw'$$

which is build according to a set of expansion rules (see next slide)

Definition (Branch, open branch and closed branch)

A branch of a tableaux is a sequence $n_1, n_2 \ldots n_k$ where n_1 is the root of the tree, n_k is a leaf, and n_{i+1} is a children of n_i for $1 \leq i < k$.

A closed branch is a branch that contains nodes marked with $w \models \phi$ and $w \not\models \phi$. All other branches are open.

If all branches are closed, the tableau is closed.

Tableau Rules for the Propositional Logic

Expansion rules for propositional connectives

$$\frac{w \models \phi \wedge \psi}{\begin{array}{c} w \models \phi \\ w \models \psi \end{array}} \qquad \frac{w \not\models (\phi \vee \psi)}{\begin{array}{c} w \not\models \phi \\ w \not\models \psi \end{array}} \qquad \frac{w \models \neg\phi}{w \not\models \phi} \qquad \frac{w \not\models \neg\phi}{w \models \phi} \qquad \frac{w \not\models (\phi \supset \psi)}{\begin{array}{c} w \models \phi \\ w \not\models \psi \end{array}}$$

$$\frac{w \models \phi \vee \psi}{w \models \phi \mid w \models \psi} \qquad \frac{w \not\models (\phi \wedge \psi)}{w \not\models \phi \mid w \not\models \psi} \qquad \frac{w \models \phi \supset \psi}{w \not\models \phi \mid w \models \psi}$$

Expansion rules for modal operators

$$\frac{w \models \Box\phi}{w' \models \phi}$$

If wRw' is already in the brench

$$\frac{w \not\models \Box\phi}{\begin{array}{c} wRw' \\ w' \not\models \phi \end{array}}$$

wher w' is new in the brench

$$\frac{w \models \Diamond\phi}{\begin{array}{c} wRw' \\ w' \models \phi \end{array}}$$

wher w' is new in the brench

$$\frac{w \not\models \Diamond\phi}{w' \not\models \phi}$$

If wRw' is already in the brench

Applications of expansion rules

- If a branch $\beta = n_1, \ldots, n_k$ contains a node n_i labelled with a premise of one of a rule ρ, and such a rule has not applied yet on this node, then ρ can be applied, and the branch is expanded in the following way
  - if ρ has only one consequence, then β is expanded in $n_1, \ldots n_k, n_{k+1}$ where n_{k+1} is labelled with the consequence of ρ
  - if ρ has two consequences (one on top of the other), then β is expanded in $n_1, \ldots n_k, n_{k+1}, n_{k+2}$ where n_{k+1} and n_{k+2} are labelled with the consequences of ρ
  - if ρ has two alternative consequences (i.e., two consequences separated by a "|"), then β is expanded into two branches $n_1, \ldots n_k, n_{k+2}$ and $n_1, \ldots n_k, n_{k+2}$, where n_{k+1} and n_{k+2} are labelled with the alternative consequences of ρ

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Example of tableaux

Example (Check satisfiability of $\Diamond(P \wedge \neg Q) \wedge \Box(P \vee Q)$)

$w \vDash \Diamond(P \wedge \neg Q) \wedge \Box(P \vee Q)$

$w \vDash \Diamond(P \wedge \neg Q)$

$w \vDash \Box(P \vee Q)$

wRw'

$w' \vDash P \wedge \neg Q$

$w' \vDash P$

$w' \vDash \neg Q$

$w' \nvDash Q$

$w' \vDash P \vee Q$

$w' \vDash P$ | $w' \vDash Q$

OPEN | CLOSED

- The tableau we have constructed starting from $w \vDash \Diamond(P \wedge \neg Q) \wedge \Box(P \vee Q)$, has an open branch (the one on the left)
- if we collect all the assertions of the form $w \vDash A$ and $w \nvDash A$ for all atomic A and the assertions of the form and wRw', which label the node of such an open branch we obtain

$$wRw', w' \vDash P, \; w' \nvDash Q$$

which corresponds to the model

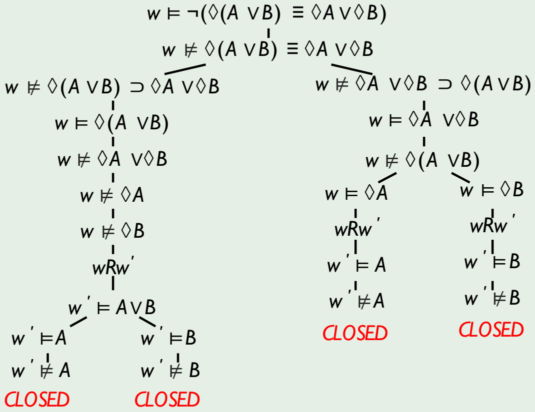
$$w \xrightarrow{R} w'$$

with A true in w' and B false in w'

Checking validity via tableaux

Example (Check validity of $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$)

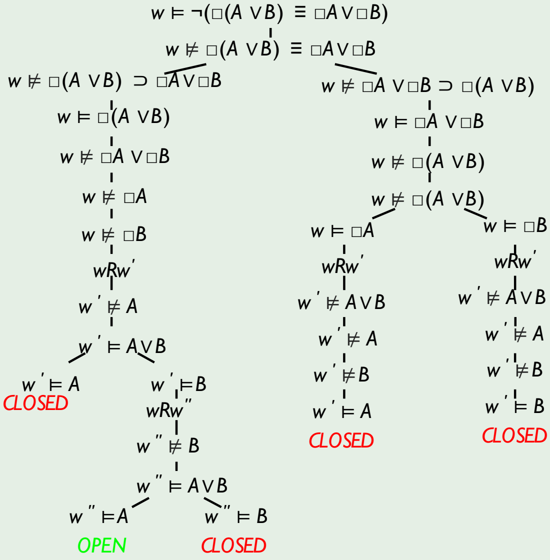
To check the validity of $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B)$, we construct a tableaux that searches for a countermodel. I.e., we check the satisfiability of $\neg(\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B)$



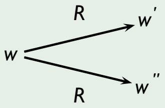
All the branches of the tableaux searching for a model of $\neg(\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B)$ are closed. This implies that there are no models for such a formulas, i.e., that there are no counter-model for $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$, and finally that $\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$, is valid.

Checking validity via tableaux

Example (Check validity of $\Box(A \vee B) \equiv \Box A \vee \Box B$)



The tableau is not closed as there is an open branch. This branch contains the statements: wRw', wRw'', $w' \not\models A$, $w' \models B$, $w'' \models A$ and $w'' \not\models B$, that correspond to the model



with A false in w', B true in w', A true in w'' and B false in w''.

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