Mathematical Logics
Modal Logic: K and more*

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Examples

- "Ann knows that P implies Q"

$$K_{Ann}(P \supset Q)$$

- "Ann does or does not know P"

$$K_{Ann}P \vee \neg K_{Ann}P$$

- "Ann knows P or knows ¬P"

$$K_{Ann}P \vee K_{Ann}\neg P$$

- "P is possible for Ann"

$L_{Ann}P$ (where L is a shorthand for $\neg K \neg$)

- "Ann knows that she knows P is possible"

$$K_{Ann}(L_{Ann}P)$$

A characterization of knowledge

- Axioms for modal **K:** conoscenza ideale
- **T**: $K\varphi \supset \varphi$ (axiom of Necessity)
  "If an agent knows that φ, then φ must be true".
  Or, . . . an agent cannot have wrong knowledge.
- **4**: $K\varphi \supset KK\varphi$ (axiom of Positive Introspection)
  "If an agent knows that φ, then (s)he knows that s(he) knows that φ".
  Or, . . . an agent knows that s(he) knows.

The logic **KT4** (better known as **S4**), gives a minimal characterization of knowledge, and corresponds to the set of reflexive and transitive frames.

But, what about ignorance? We also know what we do not know!

A characterization of knowledge (cont)

- **5**: $\neg K\varphi \supset K\neg K\varphi$ (axiom of Negative Introspection)
  "If an agent does not know that φ, then (s)he knows that s(he) does not know knows that φ". Or, . . . an agent knows that s(he) does not know.

The logic **KT45** (better known as **S5**), provides the standard characterization of knowledge, and corresponds to the set of reflexive, symmetric and transitive relations (that is, all the equivalence relations).

A characterization of belief

- Axioms for modal **K** (conoscenza perfetta)
- Agents can have false beliefs. Therefore **T** does not hold.
- $B\varphi \supset BB\varphi$ (axiom of Positive Introspection)
  "If an agent believes that φ, then (s)he believes that s(he) believes that φ".
- **5**: $\neg B\varphi \supset B\neg B\varphi$ (axiom of Negative Introspection)
  "If an agent does not believe that φ, then (s)he believes that s(he) does not know knows that φ". Or, . . . an agent believes that s(he) does not believe.

The logic **K45** provides a minimal characterization of belief, and corresponds to the set of transitive and Euclidean frames.

A characterization of belief

- Are beliefs mutually consistent? If yes then $\neg B(\varphi \wedge \neg\varphi)$ holds. (Axiom of Consistency)
  "an agent does not believe that" φ and $\neg\varphi$.
- An alternative formulation of this property is via the axiom **D**:
  $\Box\varphi \supset \Diamond\varphi$. (that is, $B\varphi \supset \neg B\neg\varphi$)
  "If an agent believes that φ then s(he) does not believe that not φ".

The logic **KD45** provides an alternative characterization of belief, and corresponds to the set of transitive, euclidean and serial relations

Note: the axiom **D** is a typical axiom of *Deontic logic.*

Exercise: Prove that $\neg B(\varphi \wedge \neg\varphi)$ is equivalent to $\Box\varphi \supset \Diamond\varphi$.

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