Mathematical Logics
Modal Logic: K and more*

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R is transitive and reflexive

The axiom 4

If a frame is reflexive and transitive then the formula

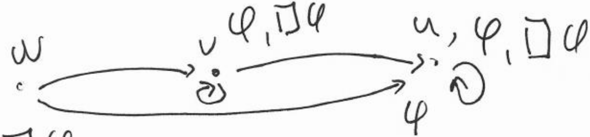
$$\mathbf{4} \quad \Box\varphi \supset \Box\Box\varphi$$

holds.

IF A FRAME F IS REFLEXIVE AND TRANSITIVE THEN

$$F \models \Box\varphi \supset \Box\Box\varphi$$

$$\underbrace{\forall w.\, R(w,w)}_{T}\;,\; \underbrace{\forall w,v,u.\big(R(w,v) \wedge R(v,u) \supset R(w,u)\big)}_{4}$$



$\Box\varphi$

[redacted]

$\Box\Box\varphi$

$\Box\varphi \supset \Box\Box\varphi$

THE LOGIC OF KNOWLEDGE WITH POSITIVE INTROSPECTION

R is transitive and reflexive - soundness

Let M be a model on a transitive frame $F = (W, R)$ and w any world in W. We prove that $M, w \models \Box\varphi \supset \Box\Box\varphi$.

1. Suppose that $M, w \models \Box\varphi$ (Hypothesis).
2. We have to prove that $M, w \models \Box\Box\varphi$ (Thesis)
3. From the satisfiability condition of $\Box$, this is equivalent to prove that for all world w' accessible from w $M, w' \models \Box\varphi$.
4. Let w' be any world accessible from w. To prove that $M, w' \models \Box\varphi$ we have to prove that for all the world w'' accessible from w', $M, w'' \models \varphi$.
5. Let w'' be a world accessible from w', i.e., $w'Rw''$.
6. From the facts wRw' and $w'Rw''$ and the fact that R is transitive, we have that

   wRw''.
7. Since $M, w \models \Box\varphi$, from the satisfiability conditions of $\Box$ we have that

   $M, w'' \models \varphi$.
8. Since $M, w'' \models \varphi$ for every world w'' accessible from w', then $M, w' \models \Box\varphi$.
9. and therefore $M, w \models \Box\Box\varphi$. (Thesis)

   Since from (Hypothesis) we have derived (Thesis), we can conclude that
10. $M, w \models \Box\varphi \supset \Box\Box\varphi$.

R is transitive and reflexive - completeness

Suppose that a frame F = (W, R) is not transitive.

1. If R is not transitive then there are three worlds w, w^{I}, $w^{II} \in W$, such that wRw^{I}, $w^{I}Rw^{II}$ but not wRw^{II}.
2. Let M be any model on F, and let φ be the propositional formula p. Let V the set p true in all the worlds of W but w^{II} where p is set to be false.
3. From the fact that w does not access to w^{II}, and that w^{II} is the only world where p is false, we have that in all the worlds accessible from w, p is true.
4. This implies that $M, \ w \vDash \Box p$.
5. On the other hand, we have that $w^{I}Rw^{II}$, and $w^{II} \vDash p$ implies that $M, w^{I} \vDash \Box\varphi$.
6. and since wRw^{I}, we have that $M, \ w \vDash \Box\Box p$.
7. In summary: $M, \ w \vDash \Box\Box p$, and $M, \ w \vDash \Box P$; from which we have that $M, \ w \vDash \Box p \supset \Box\Box p$.

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