Mathematical Logics
Modal Logic: Introduction*

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Lecture index

1. Intuition
2. Language
3. Relational structures and Satisfiability
4. Validity, unsatisfiability, Logical conseguence and equivalence

Validity relation on frames

A formula φ is valid in a world w of a frame F, in symbols
$F, w \models \varphi$ iff

$$M, w \models \varphi \text{ for all } I \text{ with } M = \langle F, I \rangle$$

A formula φ is valid in a frame F, in symbols $F \models \varphi$ iff

$$F, w \models \varphi \text{ for all } w \in W$$

If C is a class of frames, then a formula φ is valid in the class of frames C, in symbols $\models_C \varphi$ iff

$$F \models \varphi \text{ for all } F \in C$$

A formula φ is valid, in symbols $\models \varphi$ iff

$$F \models \varphi \text{ for all models frames } F$$

Logical consequence

- φ is a local logical consequence of Γ, in symbols $\Gamma \vDash \varphi$, if for every model $M = \langle F, I \rangle$ and every point $w \in W$,

$$M, w \vDash \Gamma \text{ implies that } M, w \vDash \varphi$$

- φ is a logical consequence of Γ in a class of frames C, in symbols $\Gamma \vDash_C \varphi$ if for avery model $M = \langle F, I \rangle$ with $F \in C$ and every point $w \in W$,

$$M, w \vDash \Gamma \text{ implies that } M, w \vDash \varphi$$

NOTE: Unsatisfiability and Logical equivalence defined as usual

ML Properties (same as most logics)

Proposition

A *Valid* $\longrightarrow A$ *satisfiable* $\longleftrightarrow A$ *not unsatisfiable*

A *unsatisfiable* $\longleftrightarrow A$ *not satisfiable* $\longrightarrow A$ *not Valid*

$\Gamma, A \vDash B \longleftrightarrow \Gamma \vDash A \rightarrow B$

$\Gamma \vDash \varphi \longleftrightarrow \Gamma \cup \{\neg\varphi\}$ *not satisfiable*

Proposition

if A is	*then ¬A is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

Exercises

Exercise

Show that each of the following formulas is not valid by constructing a frame $F = (W, R)$ that contains a world that does not satisfy them.

1. $\Box\bot$
2. $\Diamond\varphi \supset \Box\varphi$
3. $\Diamond\Box\varphi \supset \Box\Diamond\varphi$

Exercises

Exercise

Prove that the following formulae are valid:

- $\models \Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$
- $\models \Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$
- $\models \neg\Diamond\varphi \equiv \Box\neg\varphi$
- $\neg\Box\Diamond\Diamond\Box\Box\Diamond\Box\varphi \equiv \Diamond\Box\Box\Diamond\Diamond\Box\Diamond\neg\varphi$ (i.e., pushing in $\neg$ changes $\Box$ into $\Diamond$ and $\Diamond$ into $\Box$)

Suggestion: keep in mind the analogy $\Box/\forall$ and $\Diamond/\exists$.

Exercise

Exercise

Consider the frame $F = (W, R)$ with

- $W = \{0, 1, \ldots n-1\}$
- $R = \{(0, 1), (1, 2), \ldots, (n-1, 0)\}$

Show that the following formulas are valid in F

1. $\Box\varphi \equiv \Diamond\varphi$
2. $\varphi \equiv \underbrace{\Box \ldots \Box}_{n}\varphi$

Answers also the following questions:

3. can you explain which property of the frame R is formalized by formula 1 and 2?
4. Can you imagine another frame F', different from F that satisfies formulas 1 and 2?

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