Mathematical Logics
Modal Logic: Introduction*

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Modal logics & relational structures

The most common extensional semantics of modal logics are given in terms of relational structures.

Definition (Relational structure)

A relational structure is a tuple

$\langle W, R_{a_1}, \ldots, R_{a_n} \rangle$

where $R_{a_i} \subseteq W \times \ldots \times W$

- each $w \in W$ is called, point (world, state, time instant, situation, . . .)
- each R_{a_i} is called accessibility relation (or simply relation)

Alternative notation $\langle W, R_a \rangle_{a \in A}$

Examples of Relational structures

- Strict partial order (SPO)

  $\langle W, < \rangle$ $<$ is transitive and irreflexive[1]

- Strict linear order

  $\langle W, < \rangle$ (SPO) + for each $v \neq w \in W$, $v < w$ or $w < v$

- Partial order (PO)

  $\langle W, \leq \rangle$ $\leq$ is transitive, reflexive, and antisymmetric

- Linear order

  $\langle W, \leq \rangle$ (PO) + for each $v, w \in W$, $v \leq w$ or $w \leq v$

- Labeled transition system (LTS)

  $\langle W, R_a \rangle_{a \in A}$ and $R_a \subseteq W \times W$

- XML document

  $\langle W, R_l \rangle_{l \in L}$, W contains the components of an XML document and L is the set of labels that appear in the document

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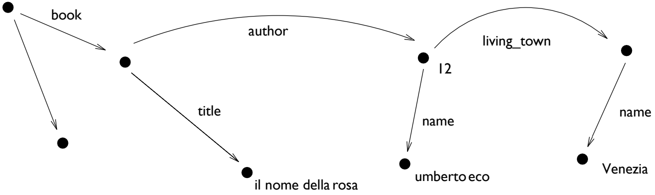
[1]Antisymmetry follows.

XML document as a relational stucture



Semantics for the basic modal logic

A basic frame (or simply a frame) is an algebraic structure

$$F = \langle W , R \rangle$$

where $R \subseteq W \times W$.

An interpretation I (or assignment) of a modal language in a frame F, is a function

$$I : P \rightarrow 2^W$$

Intuitively $w \in I(p)$ means that p is true in w, or that w is of type p.

A model M is a pair (*frame, interpretation*). I.e.:

$$M = \langle F, I \rangle$$

Satisfiability of modal formulas

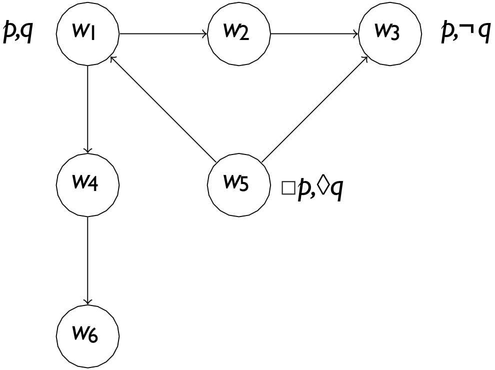
Truth is relative to a world, so we define that relation of $\models$ between a world in a model and a formula

$M, w \models p$	iff $w \in I(p)$
$M, w \models \varphi \wedge \psi$	iff $M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \varphi \vee \psi$	iff $M, w \models \varphi$ or $M, w \models \psi$
$M, w \models \varphi \supset \psi$	iff $M, w \models \varphi$ implies $M, w \models \psi$
$M, w \models \varphi \equiv \psi$	iff $M, w \models \varphi$ iff $M, w \models \psi$
$M, w \models \neg\varphi$	iff not $M, w \models \varphi$
$M, w \models \Box\varphi$	iff for all w' s.t. wRw', $M, w' \models \varphi$
$M, w \models \Diamond\varphi$	iff there is a w' s.t. wRw' and $M, w' \models \varphi$

φ is globally satisfied in a model M, in symbols,

$$M \models \varphi \quad \text{iff } M, w \models \varphi \quad \text{for all } w \in W$$

Satisfiability example



Expressing properties on structures

formula true at *w*	property of *w*
$\Diamond\top$	*w* has a successor point
$\Diamond\Diamond\top$	*w* has a successor point with a successor point
$\underbrace{\Diamond \ldots \Diamond}_{n}\top$	there is a path of length *n* starting at *w*
$\Box\bot$	*w* does not have any successor point
$\Box\Box\bot$	every successor of *w* does not have a successor point
$\underbrace{\Box \ldots \Box}_{n}\bot$	every path starting form *w* has length less then *n*

Expressing properties on structures

formula true at w	property of w
$\Diamond p$	w has a successor point which is p
$\Diamond\Diamond p$	w has a successor point with a successor point which is p
$\underbrace{\Diamond \ldots \Diamond}_{n} p$	there is a path of length n starting at w and ending at a point which is p
$\Box p$	every successor of w are p
$\Box\Box p$	all the successors of the successors of w are p
$\underbrace{\Box \ldots \Box}_{n} p$	all the paths of length n starting form w ends in a point which is p

Modalities

Modality	Symbol	Expression Symbolised
Alethic	$\Box\varphi$ $\Diamond\varphi$	it is *necessary* that φ it is *possible* that φ
Deontic	$O\varphi$ $P\varphi$ $F\varphi$	it is *obligatory* that φ it is *permitted* that φ it is *forbidden* that φ
Temporal	$G\varphi$ $F\varphi$	it will *always* be the case that φ it will *eventually* be the case that φ
Epistemic	$B_a\varphi$ $K_a\varphi$	agent *a believes* that φ agent *a knows* that φ
Contextual	$ist(c,\varphi)$	φ is *true in the context c*
Dynamic	$[\alpha]\varphi$ $(\alpha)\varphi$	φ must be true after the execution of program α φ can be true after the execution of program α
Computational	$AX\varphi$ $AG\varphi$ $AF\varphi$ $A\varphi U\vartheta$ $EX\varphi$	φ is true for every immediate successor state φ is true for every successor state φ will eventually be true in all the possible evolutions φ is true until ϑ becomes true φ is true in at least one immediate successor state

Properties of accessibility relation

Different logics are interpreted in structures of different shape

Temporal logic (logic of time): the accessibility relation is supposed to represent a temporal relation, and wRw^l means that w^l is a future world w.r.t. w *(example:ToMorrow after ToDay).Two types of structures: tree or set of paths in the direction of time for future and past. Root: the world now.*

Epistemic Logic (Logic of beliefs of agents): the accessibility relation is used to represent the beliefs of an agent *A*, and wRw^l represents the fact that w^l is a possible situation coherent with its actual situation w *(example: Fausto thinks that it is raining). Hierarchical structures with root: the world*

Description Logics (logic of Data Bases and knowledge representatio): the accessibility relation is supposed to represent a relation between two concepts (classes) *(example: the father of a person is a male). Structure is a Knowledge graph with no restrictions on shape*

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