Mathematical Logics
Modal Logic: Introduction*

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Lecture index

1. Intuition
2. Language
3. Relational structures and Satisfiability
4. Validity, unsatisfiability, Logical conseguence and equivalence

TestBooks and Readings

- *Hughes, G. E., and M.J. Cresswell (1996) A New Introduction to Modal Logic. Routledge.*
  Introductory textbook. Provides an historic perspective and a lot of explanations.
- *Blackburn, Patrick, Maarten de Rijke, and Yde Venema (2001) Modal Logic. Cambridge Univ. Press*
  More modern approach. It focuses on the formalisation of frames and structures.
- *Chellas, B. F. (1980) Modal Logic: An Introduction. Cambridge Univ. Press*

  The focus is on the axiomatization of the modal operators □ and ◇

What is Modality?

- A modality is an expression that is used to *qualify* the truth of a judgement (or, in other words, an operator that expresses a "mode" in which a proposition is true)
- It can be seen as an operator that takes a proposition and returns a more complex proposition.

Proposition	Modal Expression
John drives a Ferrari	John *is able to* drive a Ferrari
Everybody pays taxes	It is *obligatory* that everybody pays taxes

- As an example, a possible modality is expressed in natural language through **modal verbs** such as *can/could*, *may/might*, *must*, *will/would*, and *shall/should*.

What is Modality?

- In logic, modalities are formalized using an operator such as $\Box$ (and its dual $\Diamond$) that can be applied to a formula φ to obtain another formula $\Box\varphi$ ($\Diamond\varphi$).
- The truth value of $\Box\varphi$ ***is not*** a *function* of the truth value of φ (even if the truth values are related, ***it is*** a *relation*).

Example

- The fact that John is able to drive a Ferrari may be true independently from the fact that John is actually driving a Ferrari.
- The fact that it is *obligatory* that everybody pays taxes is typically true, and this is independent from the fact that everybody actually pays taxes.

Note: $\neg$ is not a modal operator since the truth value of $\neg\varphi$ is a function of the truth value of φ.

Modalities

- A modality is an expression that is used to *qualify* the truth of a judgement.
- Historically, the first modalities formalized with modal logic were the so called alethic modalities i.e.,
  1. it is possible that a certain proposition holds, usually denoted with $\Diamond\varphi$
  2. it is necessary that a certain proposition holds, usually denoted with $\Box\varphi$
- Afterwards a number of modal logics for different "*qualifications*" have been studied (see below).

Modalities

Modality	Symbol	Expression Symbolised
Alethic	$\Box\varphi$ $\Diamond\varphi$	it is *necessary* that φ it is *possible* that φ
Deontic	$O\varphi$ $P\varphi$ $F\varphi$	it is *obligatory* that φ it is *permitted* that φ it is *forbidden* that φ
Temporal	$G\varphi$ $F\varphi$	it will *always* be the case that φ it will *eventually* be the case that φ
Epistemic	$B_a\varphi$ $K_a\varphi$	agent *a* *believes* that φ agent *a* *knows* that φ
Contextual	$ist(c,\varphi)$	φ is *true in the context c*
Dynamic	$[\alpha]\varphi$ $(\alpha)\varphi$	φ must be true after the execution of program α φ can be true after the execution of program α
Computational	$AX\varphi$ $AG\varphi$ $AF\varphi$ $A\varphi U\vartheta$ $EX\varphi$	φ is true for every immediate successor state φ is true for every successor state φ will eventually be true in all the possible evolutions φ is true until ϑ becomes true φ is true in at least one immediate successor state

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