Mathematical Logics
First Order Logic*

Fausto Giunchiglia and Mattia Fumagalli

University of Trento

**Originally by Luciano Serafini and Chiara Ghidini*
Modified by Fausto Giunchiglia and Mattia Fumagalli

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Free variables

Intuition

A free occurrence of a variable x is an occurrence of x which is not bounded by a (universal or existential) quantifier.

Definition (Free occurrence)

- any occurrence of x in t_k is free in $P(t_1, \ldots, t_k, \ldots, t_n)$
- any free occurrence of x in φ or in ψ is also fee in $\varphi \wedge \psi$, $\psi \vee \varphi$, $\psi \supset \varphi$, and $\neg\varphi$
- any free occurrence of x in φ, is free in $\forall y.\varphi$ and $\exists y.\varphi$ if y is distinct from x .

Definition (Ground/Closed Formula)

A formula φ is ground if it does not contain any variable. A formula is closed if it does not contain free occurrences of variables.

Free variables

A variable x is free in φ (denote by $\varphi(x)$) if there is at least a free occurrence of x in φ.

Intuitively..

Free variables represents individuals which must be instantiated to make the formula a meaningful proposition.

- $Friends(Bob, y)$ — y free
- $\forall y . Friends(Bob, y)$ — no free variables
- $Sum(x, 3) = 12$ — x free
- $\exists x . (Sum(x, 3) = 12)$ — no free variables
- $\exists x . (Sum(x, y) = 12)$ — y free

NOTE: x is free in $P(x) \supset \forall x.Q(x)$ (the occurrence of x in red is free, the one in green is not free.

Free variable and free terms

Definition (Term free for a variable)

A term t is free for a variable x in formula φ, if and only if all the occurrences of x in φ do not occur within the scope of a quantifier of some variable occurring in t.

Example

The term x is free for y in $\exists z.hates(y, z)$. We can safely replace y with x obtaining $\exists z.hates(x, z)$ without changing the meaning of the formula.

However, the term z is not free for y in $\exists z.hates(y, z)$. In fact y occurs within the scope of a quantifier of z. Thus, we cannot substitute z for y in this sentence without changing the meaning of the sentence as we obtain $\exists z.hates(z, z)$.

Free variables and free terms - example

An occurrence of a variable x can be safely instantiated by a term free for x in a formula φ,
If you replace x with a terms which is not free for x in φ, you can have unexpected effects:
E.g., replacing x with *mother-of*(y) in the formula $\exists y.friends(x, y)$ you obtain the formula

$$\exists y.friends(mother\text{-}of(y), y)$$

Satisfiability and Validity

Definition (Model, satisfiability and validity)

An interpretation I is a model of φ under the assignment a, if

$$I \models \varphi[a]$$

A formula φ is satisfiable if there is some I and some assignment a such that $I \models \varphi[a]$.
A formula φ is unsatisfiable if it is not satisfiable.
A formula φ is valid if every I and every assignment a $I \models \varphi[a]$

Definition (Logical Consequence)

A formula φ is a logical consequence of a set of formulas Γ, in symbols $\Gamma \models \varphi$, if for all interpretations I and for all assignments a

$$I \models \Gamma[a] \quad \Rightarrow \quad I \models \varphi[a]$$

where $I \models \Gamma[a]$ means that I satisfies all the formulas in Γ under a.

Logical equivalence defined as (as usual as) bidirectional logical consequence

FOL properties - Open and Closed Formulas

- Note that for closed formulas, satisfiability, validity and logical consequence / equivalence do not depend on the assignment of variables.
- For closed formulas, we therefore omit the assignment and write I $\models \varphi$.
- More in general $I \models \varphi[a]$ if and only if $I \models \varphi[a^{\prime}]$ when $[a]$ and $[a^{\prime}]$ coincide on the variables free in φ (they can differ on all the others). This equivalence with closed formulas holds for all assignments, independent of the assignments

FOL Properties (same as PL)

Proposition

A *Valid* $\longrightarrow A$ *satisfiable* $\longleftrightarrow A$ *not unsatisfiable*

A *unsatisfiable* $\longleftrightarrow A$ *not satisfiable* $\longrightarrow A$ *not Valid*

$\Gamma, A \models B \longleftrightarrow \Gamma \models A \rightarrow B$

$\Gamma \models \varphi \longleftrightarrow \Gamma \cup \{\neg\varphi\}$ *not satisfiable*

Proposition

if A is	*then ¬A is*
Valid	*Unsatisfiable*
Satisfiable	*not Valid*
not Valid	*Satisfiable*
Unsatisfiable	*Valid*

FOL Properties of quantifiers

Proposition

The following formulas are valid

- $\forall x\, (\varphi(x) \wedge \psi(x)) \equiv \forall x \varphi(x) \wedge \forall x \psi(x)$
- $\exists x\, (\varphi(x) \vee \psi(x)) \equiv \exists x \varphi(x) \vee \exists x \psi(x)$
- $\forall x \varphi(x) \equiv \neg \exists x\, \neg \varphi(x)$
- $\forall x\, \exists x \varphi(x) \equiv \exists x \varphi(x)$
- $\exists x\, \forall x \varphi(x) \equiv \forall x \varphi(x)$

Proposition

The following formulas are not valid

- $\forall x\, (\varphi(x) \vee \psi(x)) \equiv \forall x \varphi(x) \vee \forall x \psi(x)$
- $\exists x\, (\varphi(x) \wedge \psi(x)) \equiv \exists x \varphi(x) \wedge \exists x \psi(x)$
- $\forall x \varphi(x) \equiv \exists x \varphi(x)$
- $\forall x\, \exists y \varphi(x, y) \equiv \exists y\, \forall x \varphi(x, y)$

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