Mathematical Logics
First Order Logic*

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Interpretation of terms

Definition (Assignment)

An assignment a is a function from the set of variables to the domain of interpretation Δ.

$a[x/d]$ denotes the assignment that coincides with a on all the variables but x, which is associated to d.

Example

Constants = {a,b,c}

$\Delta = \{0,2,3\}$ with I(a)= 0, I(b)= 2, I(c)= 3

If formula is

- B(x) then we have assignments a1= [0], a2=[2], a3=[3]
- A(x1,x2) then we have assignments a1= [0,0], a2= [0,2], …, a8=[3,3]
- A(x1,x2) ∧ B(x1) then we have assignments …

Interpretation of terms

Definition (Interpretation of terms)

The interpretation of a term t w.r.t. the assignment a, in symbols $\mathrm{I}(t)[a]$ is recursively defined as follows:

$$
\begin{aligned}
\mathrm{I}(x_i)[a] &= a(x_i) \\
\mathrm{I}(c_i)[a] &= \mathrm{I}(c_i) \\
\mathrm{I}(f(t_1, \ldots, t_n))[a] &= \mathrm{I}(f)(\mathrm{I}(t_1)[a], \ldots, \mathrm{I}(t_n)[a])
\end{aligned}
$$

FOL Satisfiability of formulas

Definition (Satisfiability of a formula w.r.t. an assignment)

An interpretation I **satisfies** a formula φ w.r.t. the assignment a according to the following rules:

$I \models t_1 = t_2[a]$	iff	$I(t_1)[a] = I(t_2)[a]$
$I \models P(t_1, \ldots, t_n)[a]$	iff	$\langle I(t_1)[a], \ldots, I(t_n)[a] \rangle \in I(P)$
$I \models \varphi \wedge \psi[a]$	iff	$I \models \varphi[a]$ and $I \models \psi[a]$
$I \models \varphi \vee \psi[a]$	iff	$I \models \varphi[a]$ or $I \models \psi[a]$
$I \models \varphi \supset \psi[a]$	iff	$I \not\models \varphi[a]$ or $I \models \psi[a]$
$I \models \neg \varphi[a]$	iff	$I \not\models \varphi[a]$
$I \models \varphi \equiv \psi[a]$	iff	$I \models \varphi[a]$ iff $I \models \psi[a]$
$I \models \exists x \varphi[a]$	iff	there is a $d \in \Delta$ such that $I \models \varphi[a[x/d]]$
$I \models \forall x \varphi[a]$	iff	for all $d \in \Delta$, $I \models \varphi[a[x/d]]$

Example of interpretation (as from above)

Example (Of interpretation)

Symbols

Constants: *alice*, *bob*, *carol*, *robert*
Function: *mother-of* (with arity equal to 1)
Predicate: *friends* (with arity equal to 2)

Domain

$\Delta = \{1, 2, 3, 4, \ldots\}$

Interpretation

$I(alice) = 1$, $I(bob) = 2$, $I(carol) = 3$,
$I(robert) = 2$

$$I(\textit{mother-of}) = M \qquad \begin{aligned} M(1) &= 3 \\ M(2) &= 1 \\ M(3) &= 4 \\ M(n) &= n + 1 \text{ for } n \geq 4 \end{aligned}$$

$$I(\textit{friends}) = F = \left\{ \begin{array}{lll} \langle 1,2 \rangle, & \langle 2,1 \rangle, & \langle 3,4 \rangle, \\ \langle 4,3 \rangle, & \langle 4,2 \rangle, & \langle 2,4 \rangle, \\ \langle 4,1 \rangle, & \langle 1,4 \rangle, & \langle 4,4 \rangle \end{array} \right\}$$

Example (cont'd from example above)

Exercise

Check the following statements, considering the interpretation I above:

1. $I \models Alice = Bob[a]$
2. $I \models Robert = Bob[a]$
3. $I \models x = Bob[a[x/2]]$

$I(mother\text{-}of(alice))[a] = 3$

$I(mother\text{-}of(x))[a[x/4]] = 5$

$I(friends(x, y)) =$

$x :=$	$y :=$
1	2
2	1
4	1
1	4
4	2
2	4
4	3
3	4
4	4

$I(friends(x, x)) =$

$x :=$
4

$I(friends(x, y) \wedge x = y) =$

$x :=$	$y :=$
4	4

$I(\exists x friends(x, y)) =$

$y :=$
2
1
4
3

$I(\forall x friends(x, y)) =$

$y :=$
4

NOTE: each line in a table is an assignment satisfying the formula

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